JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **27**, No. 2, May 2014
http://dx.doi.org/10.14403/jcms.2014.27.2.183

ON CONTINUED FRACTIONS, FUNDAMENTAL UNITS AND CLASS NUMBERS OF REAL QUADRATIC FUNCTION FIELDS

Pyung-Lyun Kang*

Abstract. We examine fundamental units of quadratic function fields from continued fraction of $\sqrt{D}$. As a consequence, we give another proof of geometric analog of Ankeny-Artin-Chowla-Mordell conjecture and bounds for class number, and study real quadratic function fields of minimal type with quasi-period 4.

1. Introduction

Let D be a positive square-free integer and let $u = \frac{r+s\sqrt{D}}{\sigma}$ be a fundamental unit of the real quadratic number field $\mathbb{Q}(\sqrt{D})$, where $\sigma = 2$ if $D \equiv 1 \mod 4$ and $\sigma = 1$ otherwise. Ankeny-Artin-Chowla-Mordell conjecture can be written that $s \not\equiv 0 \mod D$ when D is a prime number. Although this conjecture is checked to be true for many primes, neither a counter example nor proof is found. On the other hand, the geometric version is simpler and is proved in [12].

In this paper, we examine fundamental units of quadratic function fields from continued fraction of $\sqrt{D}$ following the similar method as in [2] for real quadratic number fields. As applications, we give another proof of geometric analog of Ankeny-Artin-Chowla-Mordell conjecture and better bounds for class numbers and study real quadratic function fields of minimal type with quasi-period 4.

Received December 11, 2013; Accepted April 16, 2014.
2010 Mathematics Subject Classification: Primary 11R58.
Key words and phrases: continued fractions, fundamental units, class number, real quadratic function field.
This work was supported by research fund of Chungnam National University.

2. Continued fractions and fundamental units: odd characteristic

Throughout this paper we fix some notations.

Let q be a power of an odd prime p, $\mathbb{A} = \mathbb{F}_q[t]$, $k = \mathbb{F}_q(t)$ and $\mathbb{A}_+$ be the subset of $\mathbb{A}$ consisting of monic polynomials. We always assume that $D \in \mathbb{A}_+$ is a squarfree monic polynomial of even degree $2d$ where $k_D := k(\sqrt{D})$, O_D the integral closure of $\mathbb{A}$ in k_D and $\epsilon_D = T_D + U_D\sqrt{D}$, $T_D, U_D \in \mathbb{A}$ the fundamental unit of k_D, i.e. the generator of $O_D^*/\mathbb{F}_q^*$.

Let $x \in k_D$ and $x = [A_0, A_1, \ldots]$ be the continued fraction of x. Define

$$Q_{-1}(x) = 0, \quad Q_0(x) = 1$$

$$Q_n(x) = A_n Q_{n-1}(x) + Q_{n-2}(x) \quad \text{for } n \geq 1$$

$$P_{-1}(x) = 1, \quad P_0(x) = A_0$$

$$P_n(x) = A_n P_{n-1}(x) + P_{n-2}(x) \quad \text{for } n \geq 1.$$

Let $B_1, \ldots, B_n$ be nonconstant polynomials in $\mathbb{A}$. Define

$$Q(\emptyset) = 1, \qquad Q(B_1) = B_1 \qquad \text{and}$$

$$Q(B_1, \ldots, B_{i+1}) = B_{i+1} Q(B_1, \ldots, B_i) + Q(B_1, \ldots, B_{i-1}).$$

For $c \in \mathbb{F}_q^*$, we say that the ordered set $\{B_1, \ldots, B_n\}$ is *c-symmetric* if $B_{n-i} = c^{(-1)^i} B_{i+1}$ for all $0 \leq i < \frac{n}{2}$. The definition of c-symmetry in [3] is incorrect. Note that if n is odd, then c must be 1. It is well-known that the continued fraction of $\sqrt{D}$ is of the form

$$[A_0 : \overline{B_1, \ldots, B_{m-1}, 2A_0/c, B_{m-1}, \ldots, B_1, 2A_0}],$$

and the fundamental unit ϵ_D of k_D for square-free D is given by

$$\epsilon_D = P_{m-1}(\sqrt{D}) + Q_{m-1}(\sqrt{D})\sqrt{D}$$

where $\{B_1, \ldots, B_{m-1}\}$ is c-symmetric, $P_{m-1} = A_0 Q(B_1, \ldots, B_{m-1}) + Q(B_2, \ldots, B_{m-1})$ and $Q_{m-1} = Q(B_1, \ldots, B_{m-1})$. Since c-symmetricity of $\{B_1, \ldots, B_{m-1}\}$ implies $Q(B_2, \ldots, B_{m-1}) = cQ(B_1, \ldots, B_{m-2})$, we have

$$T_D = A_0 Q(B_1, \ldots, B_{m-1}) + cQ(B_1, \ldots, B_{m-2}) \quad \text{and}$$

$$U_D = Q(B_1, \ldots, B_{m-1}).$$

The following theorem is given in [3], Theorem 2.1.

THEOREM 2.1. *Let m be a positive integer and $c \in \mathbb{F}_q^*$, and let $\{B_1, \ldots, B_{m-1}\}$ be a set of nonconstant polynomials in $\mathbb{A}$. Then the equation*

$$\sqrt{D} = [[\sqrt{D}], \overline{B_1, \ldots, B_{m-1}, 2[\sqrt{D}]/c, B_{m-1}, \ldots, B_1, 2[\sqrt{D}]}]$$

has infinitely many solutions $D \in \mathbb{A}_+$ if and only if $\{B_i\}$ is c-symmetric, where $[\sqrt{D}]$ denotes the polynomial part of $\sqrt{D}$. In this case,

$$D = D(X) = \alpha X^2 + \beta X + \gamma$$

with polynomial coefficients α, β and γ as X ranges over $\mathbb{F}_q[t]$, where $\alpha = 1$, $\beta = 0$ and $\gamma = c$ for $m = 1$, and, for $m > 1$,

$$\alpha = Q(B_1, \ldots, B_{m-1})^2,$$
$$\beta = 3cQ(B_1, \ldots, B_{m-2}) + (-1)^{m+1}c^2 Q(B_1, \ldots, B_{m-2})^3,$$
$$\gamma = c(cQ(B_1, \ldots, B_{m-2})^2/4 + (-1)^{m+1})Q(B_2, \ldots, B_{m-2})^2.$$

In fact,

$$
\begin{aligned}
A_0 &= [\sqrt{D}] \\
&= \frac{(-1)^{m+1}}{2} cQ(B_1, \ldots, B_{m-2})Q(B_2, \ldots, B_{m-2}) + XQ(B_1, \ldots, B_{m-1}),
\end{aligned} \tag{2.1}
$$

and

$$
\begin{aligned}
D - A_0^2 &= (-1)^{m+1} cQ(B_2, \ldots, B_{m-2})^2 + 2XcQ(B_1, \ldots, B_{m-2}) \\
&= \frac{2A_0 cQ(B_1, \ldots, B_{m-2}) + cQ(B_2, \ldots, B_{m-2})}{Q(B_1, \ldots, B_{m-1})}.
\end{aligned} \tag{2.2}
$$

Note that the discriminant $\beta^2 - 4\alpha\gamma$ of $D(X)$ is $(-1)^m 4c$, which is a unit.

Let m be a positive integer and $\{B_1, \ldots, B_{m-1}\}$ be a c-symmetric set. Define the set $S(m; B_1, \ldots, B_{m-1})$ by

$$
\begin{aligned}
S(m; B_1, \ldots, B_{m-1}) := \{&D \in \mathbb{A}_+ : D \text{ is squarefree of even degree with} \\
&\sqrt{D} = [A_0; \overline{B_1, \ldots, B_{m-1}, 2A_0/c, B_{m-1}, \ldots, B_1, 2A_0}]\}.
\end{aligned}
$$

The following theorem is the main theorem of this section.

THEOREM 2.2. *For all $D = D(X) \in S(m; B_1, \ldots, B_{m-1})$, we have $\deg U_D < \deg D$ unless*

$$X = X_0 := \left[\frac{\frac{(-1)^m}{2} cQ(B_1, \ldots, B_{m-2})Q(B_2, \ldots, B_{m-2})}{Q(B_1, \ldots, B_{m-1})} \right].$$

Proof. For $m \le 3$ it is easy to see that $\deg A_0 \ge \deg Q(B_1, \dots, B_{m-1}) = \deg U_D$ from the equation (2.1). Thus $\deg D = 2\deg A_0 > \deg U_D$ as desired. Now assume $m \ge 4$. Since $\deg D = 2\deg A_0$ and $\deg U_D = \deg Q(B_1, \dots, B_{m-1}) = \deg B_1 + \dots + \deg B_{m-1}$, it is clear from (2.1) that $\deg D \ge 2\deg U_D > \deg U_D$ unless leading terms of two parts of A_0 in (2.1) are cancelled out, i.e., X is the polynomial part of

$$\frac{\frac{(-1)^m}{2} c Q(B_1, \dots, B_{m-2}) Q(B_2, \dots, B_{m-2})}{Q(B_1, \dots, B_{m-1})},$$

which is X_0 in the Theorem. □

We say that D is of *minimal type* if $D = D(X_0)$ where X_0 is defined in Theorem 2.2.

Note that there is no minimal type for $m \le 3$.

If D is not of minimal type, then $\deg U_D \le \deg A_0$. So, we have

$$\deg T_D = \deg A_D + \deg U_D \le 2\deg A_0 = \deg D$$

as well as $\deg U_D < \deg D$. So, non-minimal type D satisfies goemetric analogue of Ackeny-Artin-Chowla-Mordell conjecture.

Goemetric analogue of Ackeny-Artin-Chowla-Mordell conjecture. *For a monic square free polynomial P of even degree, $U_P \not\equiv 0 \mod P$.*

Let μ_D be the quasi-period of $\sqrt{D}$. It is shown in the proof of Theorem 2.2 that $\deg U_D < \deg D$ for $\mu_D \le 3$.

Proposition 2.3. *Let D be a squarefree monic polynomial of even degree.*

i) *If $\mu_D \le 4$, then $\deg U_D < \deg D$.*
ii) *If $\mu_D = 5$, then $U_D \not\equiv 0 \mod D$.*

Proof. i) Suppose that $\mu_D = 4$. Then $c = 1$ since $\mu_D - 1$ is odd. Therefore

$$\sqrt{D} = [A_0, \overline{M, N, M, 2A_0, M, N, M, 2A_0}]$$

for some $M, N \in \mathbb{A}$. Let P_i/Q_i be the i-th convergent of

$$[0; \overline{M, N, cM, 2A_0/c, cM, N, M, 2A_0}].$$

Then we have

$$P_0 = 0,\ P_1 = 1,\ P_2 = N,\ P_3 = MN + 1,$$

$$Q_0 = 1,\ Q_1 = M,\ Q_2 = MN + 1,\ Q_3 = M^2 N + 2M$$

and, by (2.2),

$$D = A_0^2 + B = A_0^2 + \frac{2A_0(MN+1)+N}{M^2N+2M}.$$

Since $D \neq A_0^2$ and $D \in \mathbb{A}$, $M^2N + 2M$ divides $2A_0(MN+1)+N$. So we can write $2A_0 = MK + R$ where $K \neq 0$ and $\deg R < \deg M$. Then $B = K + G$ with $\deg G < \deg K$. Note that

$$G = \frac{R(MN+1)+N-KM}{M^2N+2M} \in \mathbb{A}.$$

Case 1. Assume $G = 0$. In this case we have

$$2A_0 = MK + R = N(RM+1) + 2R.$$

Note $R \neq 0$: for, if $R = 0$, then $2A_0 = N$ and $\mu_D < 4$. Therefore

$$\begin{aligned}\deg D &= 2\deg A_0\\ &= 2(\deg M + \deg N + \deg R) > 2\deg M + \deg N = \deg Q_3 = \deg U_D.\end{aligned}$$

Case 2. Assume $G = \frac{R(MN+1)+N-KM}{M^2N+2M} \neq 0$. Then $\deg KM \geq \deg M^2N$ since $\deg R < \deg M$. Therefore $\deg K \geq \deg M + \deg N$ and

$$\deg A_0 = \deg K + \deg M \geq 2\deg M + \deg N = \deg Q_3 = \deg U_D.$$

So, $\deg D = 2\deg A_0 > \deg U_D$.

ii) Write

$$\sqrt{D} = [A_0; \overline{M, N, N/c, cM, 2A_0/c, cM, N/c, N, M, 2A_0}].$$

Then

$$P_0 = 0,\ P_1 = 1,\ P_2 = N,\ P_3 = N^2/c + 1,\ P_4 = MN^2 + cM + N,$$

$$Q_0 = 1,\ Q_1 = M,\ Q_2 = MN+1,\ Q_3 = MN^2/c + M + N/c,$$

and

$$Q_4 = M^2N^2 + cM^2 + 2MN + 1.$$

As before, $D = A_0^2 + B$, where $B := \dfrac{2A_0(MN^2+N+cM)+N^2+c}{M^2N^2+cM^2+2MN+1}$. Since $0 \neq B \in \mathbb{A}$, we need $\deg A_0 \geq \deg M$. Again, by writing $2A_0 = MK + R$ with $\deg R < \deg M$, we can assume that $B = K + H$ with $\deg H < \deg K$, where

$$H = \frac{R(MN^2+cM+N)+N^2+c-K(MN+1)}{M^2N^2+cM^2+2MN+1}. \tag{2.3}$$

Case 1. $H = 0$: From (2.3), $H = 0 \iff K(MN+1) = R(MN^2 + cM + N) + N^2 + c = RN(MN+1) + cMR + N^2 + c$. So

$$K = RN + S, \quad \text{where } S = \frac{cRM + N^2 + c}{MN+1} \tag{2.4}$$

Thus

$$2A_0 = MK + R = M(RN+S) + R = R(MN+1) + MS. \tag{2.5}$$

We claim that $R \neq 0$ if $\mu_D = 5$. Suppose $R = 0$, then we get from (2.4) that

$$K = S = \frac{N^2 + c}{MN+1}, \text{ i.e., } N(N - MK) = K - c.$$

Since $\deg(N(M - MK)) \geq \deg N$ unless $N - MK = 0$ and $\deg(K - c) = \deg N - \deg M$, we must have $K = c$ and $N = MK = cM$, which implies that $\mu_D = 2$ since $\sqrt{D} = [A_0, \overline{M, cM}]$.

If $\deg R > 0$, then

$$\deg D = 2\deg A_0 = 2\deg(MNR) > 2\deg MN = \deg Q_4 = \deg U_D,$$

and thus $U_D \not\equiv 0 \mod D$.

Now suppose that $R = a \in \mathbb{F}_q^*$ and that $Q_4 = U_D \equiv 0 \mod D$. Since $\deg D = \deg Q_4$, $Q_4 = bD$ for some $b \in \mathbb{F}_q^*$. Note that, using (2.4), (2.5)) and our assumptions,

$$\begin{aligned} D &= A_0^2 + B \\ &= \frac{a^2M^2N^2 + a^2 + S^2M^2 + 2aM^2NS + 2aMS + 2a^2MN + 4aN + 4S}{4}. \end{aligned}$$

Thus $b = 4/a^2$ and

$$Q_4 = bD = M^2N^2 + 2MN + 1 + \frac{S^2M^2 + 2aM^2NS + 2aMS + 4aN + 4S}{a^2}.$$

Thus

$$a^2cM^2 = S^2M^2 + 2aM^2NS + 2aMS + 4aN + 4S. \tag{2.6}$$

Since $\deg S = \deg N - \deg M \geq 0$, the degree of RHS of (2.6) is $2\deg N + \deg M$, which is bigger that the degree of LHS, and we get a contradiction.

Case 2. $H \neq 0$: From (2.3), we see that

$$\deg K \geq \deg M + \deg N$$

as in i). Now we see that

$$\begin{aligned}\deg D &= 2\deg A_0\\ &= 2(\deg M + \deg K) \geq 4\deg M + 2\deg N > \deg Q_4 = \deg U_D,\end{aligned}$$

as desired. □

As a corollary of Proposition 2.3, we get another proof of geometric version of Ankeny-Artin-Chowla-Mordell conjecture when $\mu_D \leq 5$ over a field of odd characteristic.

3. Continued fractions and fundamental units: even characteristic

In this section we assume that q is a power of 2 and we consider the same problem as that of section 1 in characteristic 2. Let $\mathcal{S}$ be the set of all pairs (A, B), $A, B \in \mathbb{A}$, A nonconstant polynomial such that $X^2 + AX + B$ is irreducible and the solution $y_{(A,B)} \in \bar{k}$ generate a real quadratic extension of k. Let $\mathcal{S}'$ be the set of all pairs (A, B), $A, B \in \mathbb{A}$, A monic nonconstant polynomial such that

$$X^2 + AX + B \equiv 0 \mod C^2 \tag{3.1}$$

has no solution in $\mathbb{A}$ for each nonconstant divisor C of A. Assume that $y = y_{(A,B)}$ satisfies $|y| > 1$. It is well-known that any real quadratic function field K is of the form $K = k(y_{(A,B)}) =: k_{(A,B)}$ for some $(A, B) \in \mathcal{S}'$. It is shown in [1] that the ring of integers of $k(y)$ is $k[y]$ for $(A, B) \in \mathcal{S}'$. The problem is that different pairs in $\mathcal{S}'$ can determine the same quadratic extension. In the next lemma, we solve this problem.

Lemma 3.1. *Let $\mathcal{S}''$ be the subset of $\mathcal{S}'$ such that $\deg B < \deg A$. Then there is a one-to-one correspondence between the set of real quadratic function fields and the set $\mathcal{S}''$.*

Proof. Let $y = y_{(A,B)}$ be a zero of $X^2 + AX + B$. We may assume that $\deg B < 2\deg A$ since $X^2 + AX + B \equiv 0 \mod A^2$ has no solutions.

Suppose that $\deg B \geq \deg A$. Then there exist Q and R in $\mathbb{A}$ such that $B = AQ + R$ with $\deg R < \deg A$. Then $y' = y + Q$ is a root of $X^2 + AX + (Q^2 + R)$, and y and y' generate the same quadratic extension. Note that either $\deg(Q^2 + R) \leq \deg R < \deg A$ or $\deg(Q^2 + R) = \deg Q^2 = 2\deg B - \deg A < \deg B$ since $\deg B < 2\deg A$. One can continue this process to get $\deg B < \deg A$.

Now suppose that $k(y_{(A,B)}) = k(y_{(A',B')}) = K$ for (A, B) and $(A', B') \in \mathcal{S}''$. Since A is an invariant of K ([1], Lemma 5.2), we must have $A = A'$.

Then $B' = B + U^2 + AU$ for some $U \in \mathbb{A}$. Since $\deg B', \deg B < \deg A$, we need $\deg(U^2 + AU) = \deg U + \deg(U + A) < \deg A$, which is impossible unless $U = 0$. So, $B = B'$. □

LEMMA 3.2. *If $(A, B) \in \mathcal{S}''$, then $y = y_{(A,B)}$ is reduced, that is, $|A| = |y| > 1 > |y+A|$. In fact, $y+[y]+A$ is reduced for any $(A, B) \in \mathcal{S}$.*

Proof. This follows from the fact that the conjugate $\bar{y}$ of y is $y + A$, and that $[y] = A$ in the case $(A, B) \in \mathcal{S}''$. □

Now we see from [13], §5 that if $[y] = A$, then the continued fraction expansion of y is

$$y = [A; \overline{B_1, \ldots, B_{m-1}, A/c, B_{m-1}, \ldots, B_1, A}],$$

where $\{B_1, \ldots, B_{m-1}\}$ is c-symmetric. Imitating almost the same proof of Theorem 2.1 in [3], we get the following theorem.

THEOREM 3.3. *Let m be a positive integer and $c \in \mathbb{F}_q^*$, and let $\{B_1, \ldots, B_{m-1}\}$ be a set of nonconstant polynomials in $\mathbb{A}$. Then the equation*

$$y_{(A,B)} = [\overline{A, B_1, \ldots, B_{m-1}, A/c, B_{m-1}, \ldots, B_1}]$$

has infinitely many solutions $(A, B) \in \mathcal{S}$ if and only if $\{B_i\}$ is c-symmetric. In this case,

$$A = cQ(B_1, \ldots, B_{m-2})Q(B_2, \ldots, B_{m-2}) + XQ(B_1, \ldots, B_{m-1}) \tag{3.2}$$

and

$$B = cQ(B_2, \ldots, B_{m-2})^2 + cXQ(B_1, \ldots, B_{m-2}),$$

for $X \in \mathbb{A}$ so that A is monic.

We say that $(A, B) \in \mathcal{S}$ is of *minimal type* if (A, B) is obtained by taking

$$X = X_0 = \left[\frac{cQ(B_1, \ldots, B_{m-2})Q(B_2, \ldots, B_{m-2})}{Q(B_1, \ldots, B_{m-1})}\right],$$

that is, $\deg A < \deg Q(B_1, \ldots, B_{m-1}) = (\deg B_1 + \cdots + \deg B_{m-1})$.

Assume that $(A, B) \in \mathcal{S}''$. Let $\epsilon = \epsilon_{(A,B)} = T_{(A,B)} + U_{(A,B)}y_{(A,B)}$ be the fundamental unit. Then as in the odd characteristic case $U_{(A,B)} = Q(B_1, \ldots, B_{m-1})$. If $(A, B) \in \mathcal{S}''$ is not of minimal type, then $\deg A \geq \deg Q(B_1, \ldots, B_{m-1})$ and so

$$\deg T_{(A,B)} = \deg A + \deg Q(B_1, \ldots, B_{m-1}) \leq 2 \deg A.$$

Now Proposition 3.4 and 3.5 are characteristic 2 analog of Proposition 2.3 (i) and (ii) respectively. Let m be a positive integer and

$\{B_1, \ldots, B_{m-1}\}$ be a c-symmetric set. Define the set $T(m; B_1, \ldots, B_{m-1})$ by

$$T(m; B_1, \ldots, B_{m-1}) := \{(A, B) \in \mathcal{S}'' : y_{(A,B)} \\ = [\overline{A; B_1, \ldots, B_{m-1}, A/c, B_{m-1}, \ldots, B_1}]\}.$$

Let $\mu_{(A,B)}$ be the length of the quasi-period of $y_{(A,B)}$.

PROPOSITION 3.4. *For any* $(A, B) \in \mathcal{S}''$ *with* $\mu_{(A,B)} \leq 4$, $U_{(A,B)} \not\equiv 0 \mod A$.

Proof. If $\mu_{(A,B)} \leq 3$, then $\deg A > \deg U_{(A,B)}$ by the equation (3.2). Let $y := y_{(A,B)}$, $U := U_{(A,B)}$ and $\mu = \mu_{(A,B)}$ for simplicity. Suppose that $\mu = 4$. Since $\mu - 1$ is odd, we have $c = 1$.

$$y = [\overline{A, M, N, M}].$$

Let P_i/Q_i be the i-th convergent of $[0; \overline{M, N, M, A}]$. One can easily compute that

$$P_0 = 0,\ P_1 = 1,\ P_2 = N,\ P_3 = MN + 1,$$
$$Q_0 = 1,\ Q_1 = M,\ Q_2 = MN + 1,\ Q_3 = M^2 N$$

and

$$B = \frac{A(MN + 1) + N}{M^2 N}.$$

Since $0 \neq B = \frac{A(MN+1)+N}{M^2 N} \in \mathbb{A}$, $\deg A \geq \deg M$. Let $A = MK + R$ with $\deg R < \deg M$, and let $B = K + G$ with $\deg G < \deg K$. Then $G = \frac{R(MN+1)+N-KM}{M^2 N} \in \mathbb{A}$.

Case 1. $G = 0$: In this case we have

$$A = MK + R = N(RM + 1).$$

If $R = 0$, then $A = N$ and $\mu_D < 4$. So we assume that $R \neq 0$. Suppose that $U = M^2 N \equiv 0 \mod A$. Then $MR + 1$ divides M^2, which is impossible.

Case 2. $G \neq 0$: Since $\deg R < \deg M$ and $0 \neq G = \frac{R(MN+1)+N-KM}{M^2 N} \in \mathbb{A}$, we have $\deg KM \geq \deg M^2 N$, which implies that

$$\deg K \geq \deg M + \deg N.$$

If $\deg K > \deg M + \deg N$, then $\deg A > 2 \deg M + \deg N = \deg U$ and we are done. Assume $\deg K = \deg M + \deg N$. Then $G = a \in \mathbb{F}_q^*$. Then

$$MK = aM^2 N + R(MN + 1) + N \quad \text{and} \quad A = aM^2 N + RMN + N.$$

Suppose that $U = M^2 N \equiv 0 \mod A$. Then we must have that $0 = RMN + N = N(RM + 1)$, which is impossible. $\square$

Proposition 3.5. *Let $(A, B) \in \mathcal{S}''$ with $\mu_{(A,B)} = 5$ such that*

$$y_{(A,B)} = [\overline{A, M, N, N/c, cM, A/c, cM, N/c, N, M}].$$

If $\deg N < \deg M$, then $U_{(A,B)} \not\equiv 0 \mod A$.

Proof. From the proof of Proposition 2.3(ii), (3.1) and (3.2), it is easy to get

$$U = U_{(A,B)} = M^2N^2 + cM^2 + 1,$$
$$A = (MN^2 + cM + N)(N^2/c + 1) + X(M^2N^2 + cM^2 + 1),$$
$$B = c(N^2/c + 1)^2 + X(MN^2 + cM + N).$$

Note that both $MN^2 + cM + N$ and $(N^2/c + 1)$ are relatively prime to $(M^2N^2 + cM^2 + 1)$. Therefore $A \nmid U$ if $X = 0$. So, assume that $X \neq 0$.

If $\deg N < \frac{1}{2} \deg M$, i.e., $\deg(MN^4) < \deg(M^2N^2) = \deg U$, then $A \nmid U$ too.

Suppose that $\frac{1}{2} \deg M \leq \deg N < \deg M$ and $A = E + XU \mid U$, where $E = (MN^2 + cM + N)(N^2/c + 1)$. Then $U = YE + XYU$ and $YE = (XY + 1)U$. Since $(E, U) = 1$ and $(Y, XY + 1) = 1$, we must have

$$E = XY + 1 \quad \text{and} \quad Y = U.$$

By comparing degrees, we have $\deg X = \deg E - \deg U = 2 \deg N - \deg M \geq 0$. Having $\deg(N^2) \geq \deg M$, there exist R and S such that $N^2/c + 1 = MR + S$ with $R \neq 0$ and $\deg S < \deg M$. Then from $E = XY + 1$ and $Y = U$, we must have $X = R$ and

$$MNR + MN^2S + cMS + NS + R + 1 = 0,$$

which implies that $S \neq 0$ and $0 \leq \deg S = \deg N - \deg M$. Now from the equations of A and U, we get the result. $\square$

4. Bounds for $L(1, \chi)$

Let χ be a nonprincipal quadratic character with conductor D of degree $2d > 0$. Then it is known that

$$L(s, \chi) = \prod_{i=1}^{2d-2} (1 - \pi_i(\chi) q^{-s}),$$

with $|\pi_i(\chi)| = \sqrt{q}$, from which we have trivial bounds for $|L(1, \chi)|$;

$$(1 - \sqrt{q})^{2d-2} \leq |L(1, \chi)| \leq (1 + \sqrt{q})^{2d-2}.$$

But these bounds are not useful, and we will obtain better bounds for our purpose.

Note that

$$L(s,\chi)=\sum_{M\in\mathbb{A}_+}\frac{\chi(M)}{q^{\deg(A)s}}=\sum_{m=0}^{\infty}\chi_m q^{-ms}$$

where $\chi_m := \sum M \in \mathbb{A}_+, \deg M = m\chi(M)$ for nonnegative integer m.

It is known that $\chi_m = 0$ for $m > 2d-2$ and ([5])

$$|\chi_m| \le 2\sqrt{q}^{2d-2} = 2q^{d-1}, \quad |\chi_m| \le q^m \quad \text{for } m \le d-1. \tag{4.1}$$

Using these inequalities, we obtain the following upper bound.

Proposition 4.1. *$|L(1,\chi)| \le d + 2B < d+1$, where*

$$B=\sum_{n=1}^{d-1}\frac{1}{q^n}=\frac{q^{d-1}-1}{q^d-q^{d-1}}<\frac{1}{q-1}\le\frac{1}{2}.$$

We also have, from (4.1),

Lemma 4.2. *For $|s-2| \le 4/3$, we have*

$$|L(s,\chi)| \le 2q^{d-1} < q^d = |D|^{1/2}$$

for all d for $q \ge 5$, $d \ge 2$ for $q = 3$, and $d \ge 5$ for $q = 2$.

Due to the above Lemma and Lemma 11.7 in [9], we have

Theorem 4.3. *For $0 < \epsilon \le 4/27$, we have*

$$L(1,\chi) \ge \frac{\epsilon/16}{|D|^{\epsilon/2}}$$

for all d for $q \ge 5$, $d \ge 2$ for $q = 3$, and $d \ge 5$ for $q = 2$.

Proof. From Lemma 11.7 in [Wa], $L(1,\chi) \ge \frac{1}{4}(1-\alpha)(|D|^{1/2})^{-4(1-\alpha)}$ for $26/27 \le \alpha < 1$. By taking $4(1-\alpha) = \epsilon$, we get our lower bound of $L(1,\chi)$. □

5. Yokoi's invariants in odd characteristic

Let D be a square-free monic polynomial of even degree $2d$. Let $\epsilon_D = T_D + U_D\sqrt{D}$ be the fundamental unit of K_D with $\alpha = N(\epsilon_D) = 1$ where γ a fixed generator of $\mathbb{F}_q^*$. Let N_D and A_D be the unique polynomials such that

$$T_D = U_D^2 N_D + A_D,$$

with $A_D = 0$ or $\deg A_D < \deg U_D^2$. We call N_D the *Yokoi invariant* of D (cf.[10], [11]). Since

$$DU_D^2 = T_D^2 - \alpha = U_D^4 N_D^2 + 2A_D U_D^2 N_D + A_D^2 - \alpha,$$

there exists a unique B_D such that $A_D^2 - \alpha = B_D U_D^2$. Then

$$D = U_D^2 N_D^2 + 2A_D N_D + B_D = T_D N_D + A_D N_D + B_D. \tag{5.1}$$

Note that if $\deg U_D > 0$, then A_D cannot be 0 since $A_D^2 - \alpha = B_D U_D^2$. If $A_D \neq 0$, then $B_D = 0$ when $A_D^2 = \alpha$, or $\deg B_D = 2\deg A_D - 2\deg U_D < \deg A_D$.

Lemma 5.1. *We have $N_D = [D/T_D]$, where $[x]$ denotes the polynomial part of x. Moreover, if $A_D = 0$, then $D = (\beta T_D)^2 - \beta^2\alpha$, that is, D is of Chowla type.*

Proof. Suppose that $N_D = 0$, that is, $\deg T_D < 2\deg U_D$. Since $T_D^2 - DU_D^2 = \alpha \in \mathbb{F}_q^*$, $\deg D = 2\deg T_D - 2\deg U_D = \deg T_D + (\deg T_D - 2\deg U_D) < \deg T_D$, so $[D/T_D] = 0$.

Assume now that $N_D \neq 0$, that is, $\deg T_D \geq 2\deg U_D$. Then from (5.1) $[D/T_D] = N_D$ since $\deg T_D = 2\deg U_D + \deg N_D > \deg A_D + \deg N_D$ and $\deg B_D < \deg A_D < \deg T_D$ (unless $B_D = 0$).

If $A_D = 0$, then $U_D \in \mathbb{F}_q^*$ and $D = T_D^2\beta^2 - \alpha\beta^2$ for some $\beta \in \mathbb{F}_q^*$. $\square$

In the proof of Lemma 5.1, we also observed the followings.

(1) The three conditions are equivalent:
i) $N_D = 0$ ii) $\deg D < \deg T_D$ iii) $\deg T_D < 2\deg U_D$.

(2) $A_D = 0$ holds only if $\deg U_D = 0$.

Therefore if $N_D \neq 0$, $\deg D \geq \deg T_D \geq 2\deg U_D$. Thus $U_D \not\equiv 0 \mod D$ since $\deg U_D < \deg D$. This is another proof of geometric version of Ankeny-Artin-Chowla conjecture.

Using Proposition 4.1, Theorem 4.3 and the fact that

$$L(1,\chi) = \frac{q-1}{\sqrt{|D|}} h_D R_D,$$

where h_D is the ideal class number of O_D and $R_D = \log|\epsilon_D|$ is the regulator of O_D, we get the bound for h_D in the following.

Theorem 5.2. *Let D be a monic square-free polynomial of even degree and N_D be Yokoi invariant of D explained before. Then*

(1) $\deg \epsilon_D = \begin{cases} \deg D - \deg N_D & \text{if } N_D \neq 0 \\ \deg T_D > \deg D & \text{if } N_D = 0 \end{cases}$.

(2) *If* $N_D \neq 0$, *then*

$$\frac{\epsilon}{16}\frac{q^{d(1-\epsilon)}}{(q-1)(2d-\deg N_D)\log q} < h_D < \frac{q^d(d+1)}{(q-1)(2d-\deg N_D)\log q}$$

for $0 < \epsilon < 4/27$. *For left inequality, we need that* $d \geq 2$ *for* $q = 3$ *and* $d \geq 5$ *for* $q = 2$.

Note that $\deg T_D < \deg U_D^2$ if $N_D = 0$. We now can derive the similar bound for h_D by using polynomial part $\left[\frac{U_D^2}{T_D}\right]$ of $\frac{U_D^2}{T_D}$ instead of $\left[\frac{D}{T_D}\right]$. For this, we write $T_D = DM_D + E_D$ where $\deg E_D < \deg D$. Note $E_D = 0$ cannot happen since $T_D^2 \equiv E_D^2 \equiv \alpha \mod D$. We now claim

$$\left[\frac{U_D^2}{T_D}\right] = M_D. \tag{5.2}$$

For, if $M_D = 0$, then $\deg T_D < \deg D$, $\deg U_D^2 = \deg T_D^2 - \deg D = \deg T_D + (\deg T_D - \deg D) < \deg T_D$. Therefore if $M_D = 0$, then $\left[\frac{U_D^2}{T_D}\right] = 0$. Suppose $M_D \neq 0$. Then, if one writes $E_D^2 - \alpha = DF_D$,

$$U_D^2 = DM_D^2 + 2E_DM_D + F_D = m_DT_D + E_DM_D + F_D.$$

Again by degree computation, $\left[\frac{U_D^2}{T_D}\right] = M_D$.

Note that if D is not of minimal type, then $M_D = 0$ or $m_D = \deg M_D = 0$.

THEOREM 5.3. *Let* P *be a monic prime of even degree. Then Ankeny-Artin-Chowla-Mordell conjecture is true if and only if* $E_PF_P \not\equiv 2\alpha M_P \mod P$. *In particular, if* $M_P \equiv 0 \mod P$, *then Ankeny-Artin-Chowla-Mordell conjecture is true.*

Proof. Suppose that $M_P \equiv 0 \mod P$. Note that $E_P \not\equiv 0 \mod P$ and $\alpha = \gamma$, which is not a square in $\mathbb{F}_q$. Thus $F_P \not\equiv 0 \mod P$. The others are straightforward. □

The following Theorem is analog of Theorem 5.2 using M_D in equation (5.2). Note that from the definition that $\deg N_D > 0$ (resp. $N_D = 0$) if and only if $M_D = 0$ (resp. $\deg M_D > 0$).

THEOREM 5.4. *For any monic square-free polynomial* D *of even degree* $2d > 0$,

(1) $[\epsilon_D/D] = 2M_D$

(2) *If $M_D \neq 0$, then*

$$\frac{\epsilon q^{d(1-\epsilon)}}{16(q-1)(2d+\deg M_D)\log q} < h_D < \frac{q^d(d+1)}{(q-1)(2d+\deg M_D)\log q}.$$

where $0 < \epsilon < 4/27$. For left inequality, we need that $d \geq 2$ for $q=3$ and $d \geq 5$ for $q=2$.

Proof. (1) Let $[U_D\sqrt{D}/D] = M'$, that is, $U_D\sqrt{D} = M'D + b$ with $|b| < |D|$. Then

$$1 > |T_D - U_D\sqrt{D}| = |(M_D - M')D + (E_D - b)|.$$

We must have $M' = M_D$ and $[\epsilon_D/D] = 2M_D$.

(2) follow from Proposition 4.1, Theorem 4.3 and (1). □

A solution (X, Y) of $X^2 - DY^2 = \beta Z$ with monic Z and $\beta \in \mathbb{F}_q^*$ is said to be trivial if $Z = M^2$ and M divides both X and Y.

Lemma 5.5. *If there is a nontrivial solution to $X^2 - DY^2 = \beta Z$, then*

$$\deg Z \geq \deg N_D.$$

Proof. With the notations of §1, we see that

$$n_D = \deg N_D = d - (\deg B_1 + \cdots + \deg B_{m-1}).$$

By Lemma 1.24 and 1.25 of [3],

$$\deg Z = d - \deg B_i,$$

for some $0 < i < m$. Hence we get the result. □

Now following the arguments of [7], §3, and using [1] Proposition 4.1, we get

Theorem 5.6. (1) *Let p_D be the least degree of primes which splits in $k(\sqrt{D})$. If $n_D = \deg N_D \neq 0$, then $h_D \geq n_D/p_D$.*

(2) *If $n_D \geq d-1$ and $h(D) = 1$, then D is Richard-Degret type.*

(3) *Let p_D be the least degree of primes which is noninert in $k(\sqrt{D})$. If $n_D = \deg N_D \neq 0$ and h_D is odd, then $h_D \geq n_D/p_D$.*

6. Real quadratic function fields of minimal type with qusi-period 4

We have the following analogue of Siegel's theorem, which follows easily from Theorem 7.6.3 of [8].

PROPOSITION 6.1. *Let D be a monic square-free polynomial of even degree. We have*

$$\lim_{\deg D \to \infty} \frac{\log(h_D \deg \epsilon_D)}{\log |D|} = \frac{1}{2}.$$

LEMMA 6.2. *Let $\{D_n\}$ be a sequence of monic square-free polynomials such that $\deg D_n \to \infty$ as $n \to \infty$ and $\deg D_n \leq \deg D_{n+1}$. Assume that $M_{D_n} \neq 0$, and that the sequence $\{m_{D_n} = \deg M_{D_n}\}$ is bounded. Then the sequence $\{h_{D_n}\}$ is not bounded.*

Proof. By Theorem 5.4(1),

$$\deg \epsilon_{D_n} = \deg D_n + \deg M_{D_n},$$

and so

$$\lim_{n \to \infty} \frac{\log \deg \epsilon_{D_n}}{\log |D_n|} = 0.$$

If $\{h_{D_n}\}$ is bounded, then we get a contradiction from Proposition 6.1. □

Now we construct a family of monic polynomials of minimal type with quasi-period 4.

PROPOSITION 6.3. *Let B be a nonconstant polynomial. For any nonzero polynomials E and F with $\deg F < \deg B$, we put*

$$\begin{aligned} D = D(E,F) &:= \frac{1}{4}(B^2EF - BE + BF^2 + F)^2 - BEF + E - F^2 \\ &= \frac{1}{4}(B(BF-1))^2E^2 + \frac{1}{2}((BF)^2 - 1)(BF-2)E + \frac{1}{4}(BF-1)^2F^2. \end{aligned}$$

Then D is of minimal type with period and quasi-period 4 and

$$\begin{aligned} \sqrt{D} =& [-(B^2EF - BE + BF^2 + F)/2, \\ & \overline{B, BEF, B, -(B^2EF + BE + BF^2 + F)}]. \end{aligned}$$

Furthermore, if D is square-free, then $M_D = -2([B/F] + [1/F^2] + 2[1/E])$.

LEMMA 6.4. (Nagell, Theorem 45) *Let $f(X) = \alpha X^2 + \beta X + \gamma$ be a quadratic polynomial in $\mathbb{A}[X]$ with α monic. For each integer t, there exist infinitely many irreducibles P which is a divisor of $f(T)$ with some polynomial T with degree $\geq t$*

Proof. The proof is exactly the same as the case of $\mathbb{Z}$. □

Assuming Lemma 6.4, we can prove the following analogue of [6], Proposition 6.1.

Proposition 6.5. *Let $f(X)$ be as in Lemma 6.4. Let t_1 be a positive integer such that for any A with $\deg A \geq t_1$, the leading coefficient of $f(A)$ is a square in $\mathbb{F}_q^*$. Suppose that the discriminant $d(f) = \beta^2 - 4\alpha\gamma$ is not 0 and that $\gcd(\alpha, \beta, \gamma)$ is square-free. Then, the set $\{f(A) : \deg A \geq t_1\}$ contains infinitely many square-free elements.*

Proof. For any real number $x > t_1$, define

$$A(x) := \#\{A \in \mathbb{A} |\, t_1 \leq \deg A \leq x,\ f(A) \text{ is square-free}\}.$$

Our aim is to prove $A(x) \to \infty$ as $x \to \infty$. By Lemma 6.4, there exist infinitely many monic irreducibles P which is a divisor of $f(A)$ for some A with $\deg A \geq t_1$. We arrange these $P_1, P_2, \ldots,$ so that $\deg P_i \leq \deg P_{i+1}$. As

$$\sum_i \frac{1}{|P|^2} < \sum_i \frac{1}{q^i} = \frac{q}{q-1} < \infty,$$

there is a number $m \geq 2$ such that

$$\sum_{i=m}^{\infty} \frac{1}{|P_i|^2} < \frac{q-1}{2q}, \tag{6.1}$$

and

$$P_i \not| \alpha \cdot d(f) \quad \text{if } i \geq m.$$

Put

$$\mathbf{P} := P_1^2 \cdots P_{m-1}^2.$$

As in the proof of Proposition 6.1, [6], there exists $A_{0,i} \in \mathbb{A}$ such that $\operatorname{ord}_{P_i}(f(A_{0,i})) < 2$ for each i $(1 \leq i \leq m-1)$ and $A_0 \in \mathbb{A}$ with $\deg A_0 > t_1$ such that

$$T_0 \equiv A_{0,i} \mod P_i^2, \quad \text{for } 1 \leq i \leq m-1.$$

Consider a quadratic polynomial

$$g(Y) := f(\mathbf{P}Y + A_0) \in \mathbb{A}[Y].$$

Define, for a positive real number $y > \deg A_0 - \deg \mathbf{P}$,

$$B(y) := \#\{A \in \mathbb{A} :\ \deg A_0 - \deg \mathbf{P} < \deg A \leq y,\ g(A) \text{ is square-free}\}.$$

Then, for $y > \deg T_0 - \deg \mathbf{P}$,

$$A(\deg \mathbf{P} + y) \geq B(y).$$

For a monic irreducible P and a real number $y > \deg T_0 - \deg \mathbf{P}$, we define

$$\hat{B}_P(y) := \#\{A \in \mathbb{A} : \deg A_0 - \deg \mathbf{P} < \deg A \leq y,\ g(A) \equiv 0 \mod P^2\}.$$

Then we have

$$B(y) \geq (q^y - q^{\deg A_0 - \deg \mathbf{P}}) - \sum_P \hat{B}_P(y).$$

(I) If P is different from P_i's, $i \geq 1$, then P does not divide $f(A)$ for all A with $\deg A \geq t_1$. Hence P does not divide $g(A)$ for all A with $\deg A > \deg A_0 - \deg \mathbf{P}$. Thus $\hat{B}_P(y) = 0$ in this case.

(II) Suppose that $P = P_i$ for some $1 \leq i \leq m-1$. Since $\mathbf{P}A + T_0 \equiv A_0 \equiv A_{0,i} \mod P_i^2$, we see that $g(A) \equiv f(A_{0,i}) \not\equiv 0 \mod P_i^2$. Hence $\hat{B}_P(y) = 0$ in this case, too.

Let $G = \gcd(\alpha, \beta, \gamma)$, and

$$f(X) = G \prod_{k=1}^{\nu} f_k(X),$$

be the factorization of $f(X)$ into irreducible polynomials in $\mathbb{A}[X]$ ($\nu = 1$ or 2). Then

$$g(Y) = G \prod_{k=1}^{\nu} g_k(X), \quad g_k(Y) := f_k(\mathbf{P}Y + A_0).$$

Let $n_k = \deg f_k(X)$. Then there are some real numbers $y_k > 0$ and $c_k > 0$ such that

$$\deg A \geq y_k \Rightarrow \deg g_k(A) < c_k + n_k \deg A,$$

$$\deg A \geq y_k, \deg A \geq \deg B \geq \deg A_0 - \deg \mathbf{P} \Rightarrow \deg g_k(B) \leq \deg g_k(A).$$

Put $y_0 := \max\{y_k\}$ and $c := \max\{c_k\}$, and assume that $y \geq y_0$.

(III) Suppose that $P = P_i$ and $\deg P_i > c + y$ with some $i \geq m$. Let $T \in \mathbb{A}$ with $\deg T_0 - \deg \mathbf{P} \leq \deg T \leq y = \deg A$, we have

$$\deg g_k(T) \leq \deg g_k(A) < c_k + n_k \deg A \leq n_k(c_k + \deg A) < n_k \deg P.$$

As $n_k \leq 2$ and $g_k(T) \neq 0$, $g_k(T)$ cannot be divisible buy P^2. If $\nu = 1$, we are done.

Now assume that $\nu = 2$. Similar process as in [6] will give the result.

(IV) Suppose that $P = P_i$ and $\deg P_i \leq c + y$ with $i \geq m$. If $d \geq 2 \deg P$, then for each residue class modulo P^2, there are $(q-1)q^{d-2\deg P}$ elements of $\mathbb{A}_d$. Let $\ell_i := \max\{2 \deg P_i, \deg A_0 - \deg \mathbf{P}\}$. Then we can see that

$$\hat{B}_P(y) \leq 1 + (q^{y-\ell+1} - 1) \sum_{\substack{A \mod P^2 \\ g(A) \equiv 0 \mod P^2}} 1 \leq 2q^{y-\ell_i+1}.$$

Now, using (I)-(IV) and (6.1), we have

$$
\begin{aligned}
B(y) &\geq (q^{y+1} - q^{\deg A_0 - \deg \mathbf{P}}) - \sum_{i\geq m,\, \deg P_i \leq c+y} B_{P_i}(y) \\
&\geq (q^{y+1} - q^{\deg A_0 - \deg \mathbf{P}}) - \sum_{i\geq m,\, \deg P_i \leq c+y} 2q^{y-\ell_i+1} \qquad (6.2)\\
&\geq q^{y+1}(1 - \frac{q-1}{q}) - q^{\deg A_0 - \deg \mathbf{P}}.
\end{aligned}
$$

The last term in (6.2) tends to ∞ as $y \to \infty$, and this proves our proposition. □

Theorem 6.6. *Let B be a nonconstant monic polynomial with $B+1$ square-free. Then for any positive integer h, there exist infinitely many real quadratic function fields $k(\sqrt{D})$ with period and quasi-period 4 of minimal type such that $h_D > h$ and $M_D = 2(B-1)$.*

Proof. In Proposition 6.3, we take $F = -1$ and $\deg E > 0$. Then, if D is square-free, then $M_D = 2B+2$. Taking $2V = E$, we easily see that

$$D(V) = B^2(B+1)^2V^2 - (B^2-1)(B+2)V + \frac{(B+1)^2}{4}.$$

Also it is easy to see that $\gcd(B^2(B+1)^2, (B^2-1)(B-2), (B+1)^2) = B+1$, which is square-free by assumption. Thus the quadratic polynomial $D(V)$ satisfies the conditions of Proposition 6.5. Now apply Proposition 6.5 and Lemma 6.2 to get the result. □

7. Yokoi's invariants in even characteristic

In this section we consider Yokoi's invariants in even characteristic.

We have the following variant of Siegel's theorem in characteristic 2, which also follows from Theorem 7.6.3 of [8].

Proposition 7.1. *Let $(A,B) \in \mathcal{S}''$. We have*

$$\lim_{\deg A \to \infty} \frac{\log(h_{(A,B)} \deg \epsilon_{(A,B)})}{\log |A|} = 1.$$

Let $(A,B) \in \mathcal{S}''$ and $y = y_{(A,B)}$ be as in §2. Let y' be the conjugate of y and

$$\epsilon = T + Uy = T' + U'y'$$

be the fundamental unit of $K_{(A,B)}$. We define Yokoi's invariants $N = N_{(A,B)}$ and $M = M_{(A,B)}$ by

$$T' = (U')^2 N + E \quad \text{and} \quad (U')^2 = T'M + F$$

with $\deg E < \deg (U')^2$ and $\deg F < \deg T'$.

REMARK 7.2. The reason for using y' instead of y is that y is reduced, but y' is not as $\sqrt{D}$. If we use y, then we always have $N = 0$. Note also that $U' = U$.

PROPOSITION 7.3. *Let the notation be as above. Then*

$$N = \left[\frac{A}{U'}\right] = \left[\frac{A^2}{T'}\right] \quad \textit{and} \quad M = \left[\frac{U'}{A}\right] = \left[\frac{T'}{A^2}\right].$$

Proof. Since $yy' = B$ and $|y'| < 1$, we easily see that $|T' + U'A| = |U'y'| < |U'|$. Thus $T' = U'A + V$ with $\deg V < \deg U'$, which implies that $N = [A/U']$. Let $A = U'N + W$ with $\deg W < \deg U'$. Then

$$A^2 = A(U'N + W) = (AU' + V)N + AW + VN = T'N + AW + VN,$$

and it is easy to see that $\deg AW < \deg AU = \deg T'$ and $\deg VN = \deg V + \deg A - \deg U < \deg A \leq \deg AU = \deg T'$. Thus $N = [A^2/T]$.

Let $T' + U'A = V$ with $\deg V < \deg U$. We have, since the polynomial part function $[\cdot]$ is additive, $[(U')^2/T'] = [U'/A] + [U'V/AT']$ and $[T'/A^2] = [U'/A] + [V/A^2]$. Now one can show easily that $\deg U'V < \deg AT$ and $\deg V < \deg A^2$, which implies the result for M. □

Let $\chi_{(A,B)}$ be the quadratic character for $k_{(A,B)}/k$. Then it is known that the conductor of $\chi_{(A,B)}$ is A^2 ([4]). Then as in the odd characteristic case we have;

THEOREM 7.4. *Let the notation be as above. Let $d = \deg A$. Then*

i) $\deg \epsilon_{(A,B)} = \begin{cases} 2\deg A - \deg N & \textit{if } N \neq 0 \\ 2\deg A + \deg M & \textit{if } M \neq 0 \end{cases}$

ii) *If $N \neq 0$, then*

$$h_{(A,B)} < \frac{q^d(d+1)}{(q-1)(2d - \deg N)\log q}.$$

If $\deg M \neq 0$, then

$$h_{(A,B)} < \frac{q^d(d+1)}{(q-1)(2d + \deg M)\log q}.$$

iii) *Let* $0 < \epsilon < 4/27$. *Assume that* $d \geq 2$ *for* $q = 3$ *and* $d \geq 5$ *for* $q = 2$. *If* $N \neq 0$, *then*

$$h_{(A,B)} > \frac{\epsilon q^{d(1-\epsilon)}}{16(q-1)(2d - \deg N)\log q}.$$

If $M \neq 0$, *then*

$$h_{(A,B)} > \frac{\epsilon q^{d(1-\epsilon)}}{16(q-1)(2d + \deg M)\log q}.$$

Proposition 7.5. *Let the notation be as above. Let* C *be a nonconstant polynomial. For any nonzero polynomials* E *and* F *with* $\deg F < \deg C$, *we put*

$$A := (CE+F)(CF+1) \quad \text{and} \quad B := (CF+1)E+F^2 = (CE+F)F+E.$$

Then (A, B) *is of minimal type with period and quasi-period* 4 *and*

$$y_{(A,B)} = [\overline{A, C, CE+F, C}] \quad \text{and} \quad y'_{(A,B)} = [0, \overline{C, CE+F, C, A}].$$

Moreover, if $(A, B) \in \mathcal{S}''$, *then the Yokoi's invariant* $M = [C/F] + [1/E]$.

Now the problem is to show that there exist infinitely many pairs (E, F) such that $((CE+F)(CF+1), (CE+F)+E)\mathcal{S}''$. Suppose that $F = 1$ and $P = C+1$ is irreducible.

Lemma 7.6. *Let* $P = C+1$ *be irreducible. If* $Q = CG^2+1$ *with* $\deg G > 0$ *is irreducible, then* $(A, B) \in \mathcal{S}''$ *for* $A = (CG^2+1)(C+1) = PQ$ *and* $B = CG^2+1+G^2 = PG^2+1 = Q+G^2$.

Proof. We need to check that

$$X^2 + AX + B \equiv 0 \mod H^2$$

has no solution for $H = P$ or Q. Suppose that we have a solution x for $H = Q$. Then $x \equiv G \mod Q$. Write $x = G + QL$. Then

$$G^2 + PQG + Q + G^2 \equiv 0 \mod Q^2,$$

that is, $PG+1 \equiv 0 \mod Q$, which is impossible, since $\deg Q > \deg(PG+1)$. Similarly we get the result for $H = P$. □

Corollary 7.7. *Let* C *be nonconstant polynomial with* $P = C+1$ *irreducible. If Bunyakovsky's conjecture for* $f(X) = CX^2+1$ *is true, then there exist infinitely many real quadratic function fields* $K_{(A,B)}$ *with period and quasi-period* 4 *of minimal type such that* $h_{(A,B)} > h$ *and* $M = C$.

References

[1] S. Bae, *Real quadratic function fields of Richaud-Degert type with ideal class number one,* Proc. AMS **140** (2012), 403-414.
[2] D. Byeon and S. Lee, *A note on units of real quadratic fields,* Bull. Korean Math. Soc. **49** (2012), 767-774.
[3] C. Friesen, *Continued fractions and real quadratic function fields,* Doctoral Thesis, Brown University (1989).
[4] R. Hashimoto, *Ankeny-Artin-Chowla conjecture and continued fraction,* J. Number Th. **90** (2001), 143-153.
[5] C. N. Hsu, *Estimates for coefficients of L-functions for function fields,* Finite Fields and their Appl. **5** (1999), 76-88.
[6] F. Kawamoto and K. Tomita, *Continued fractions and certain real quadratic fields of minimal type,* J. Math. Soc. Japan **60** (2008), 865-903.
[7] R. A. Mollin and H. C. Williams, *A complete generalization of Yokoi's p-invariants,* Colloquium Mathematicum **63** (1992), 285-294.
[8] G. D. Villa Salvador, Topics in the theory of algebraic function fields, Birkhäuser (2006).
[9] L. Washington, *Introduction to cyclotomic fields* GTM vol 83 Springer-Verlag, 1982.
[10] H. Yokoi, *The fundamental unit and bounds for class numbers of real quadratic fields,* Nagoya Math. J. **124** (1991), 181-197.
[11] H. Yokoi, *New invariants and class number problem in real quadratic fields,* Nagoya Math. J. **132** (1993), 175-197.
[12] J. Yu and J. K. Yu, *A note on a geometric analogue of Ankeny-Artin-Chowla's conjecture,* Contemporary Math. A. M. S. **210** (1998), 101-105.
[13] R. Zuccherato, *The continued fraction algorithm and regulator for quadratic function fields of characteristic 2,* J. Algebra **190** (1997), 563-587.

*
Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: `plkang@cnu.ac.kr`